Some Special Cases of Non-Amorphous Association Schemes with A.V. Ivanov's Condition (Algebraic Combinatorics)

Ikuta, Takuya

数理解析研究所講究録 1327 巻 2003 年 76-85

Some Special Cases of Non-Amorphous Association Schemes with A.V.Ivanov's Condition

神戸学院女子短期大学　Kobe Gakuin Women's College
生田 卓也　Takuya Ikuta

In this paper, we mention the following two topics:

Firstly, we study one of the generalizations of *Edwin van Dam*'s result. In his paper, he considered strongly-regular decompositions of the complete graph. He found the matrix form of the 1st eigenmatrix P of non-amorphous association schemes of class $d=4$. We generalize his results and study a combinatorial structure of the 1st eigenmatrix P of non-amorphous association schemes with some condition.

In general, the special cases of the 1st eigenmatrix P of non-amorphous association schemes with some condition have relation to the incidence matrix of a ***Balanced Incomplete Block Design.***

The second is a joint work with *Akihiro Munemasa* (Kyushu University). We present a new example of non-amorphous association schemes over the finite field. And, we mention possible new infinite series containing our new example different from van Dam's infinite series.

§1. Preliminaries

Let $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ be a symmetric association scheme of class d over a finite set X of cardinality n. We refer [2] for notations and general theory of association schemes.

Let $P=(p_{i,j})_{\substack{0\le i\le d\\0\le j\le d}}$ and $Q=(q_{i,j})_{\substack{0\le i\le d\\0\le j\le d}}$ be the 1st and 2nd eigenmatices of $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ respectively.

Let $\{\Lambda_j\}_{0\le j\le d'}$ be a partition of $\{0,1,\dots,d\}$ with $\Lambda_0=\{0\}$. We define $R_{\Lambda_j}=\bigcup_{\ell\in\Lambda_j}R_\ell$. If $\tilde{\mathcal{X}}=(X,\{R_{\Lambda_j}\}_{0\le j\le d'})$ satisfies the conditions of association schemes, then we call $\tilde{\mathcal{X}}=(X,\{R_{\Lambda_j}\}_{0\le j\le d'})$ a ***fusion*** scheme of $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$. $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ is called ***amorphous*** if $\tilde{\mathcal{X}}=(X,\{R_{\Lambda_j}\}_{0\le j\le d'})$ is a fusion scheme of $\mathcal{X}=(X,\{R_i\}_{0\le i\le d})$ for any partition $\{\Lambda_j\}_{0\le j\le d'}$ with $\Lambda_0=\{0\}$.

There is a simple criterion in terms of P for a given partition $\{\Lambda_j\}_{0\le j\le d'}$ to give rise to a fusion scheme (due to E.Bannai [1], M.Muzychuk [9]): There exists a partition $\{\Delta_j\}_{0\le j\le d'}$ of $\{0,1,\dots,d\}$ with $\Delta_0=\{0\}$ such that each (Δ_i,Λ_j)-block of P has a constant row sum. The constant row sum turns out to be the (i,j) entry of the 1st eigenmatrix of the fusion scheme.

In terms of the 2nd eigenmatrix Q, the above criterion becomes the following:

There exists a partition $\{\Delta_j\}_{0\le j\le d'}$ of $\{0,1,\dots,d\}$ with $\Delta_0=\{0\}$ such that each (Λ_i,Δ_j)-block of Q has a constant row sum.

In [5] Edwin R. van Dam considered strongly-regular demposition of the complete graph. The following result is due to him.

Theorem 1 *(Edwin R. van Dam) Let* $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ *be a commutative strongly-regular decomposition of the complete graph on* X. *Let* (X, Γ_i) *have valency* k_i *and restricted eigenvalues* r_i *and* s_i *(where we do not assume that* $r_i > s_i$*), Then* $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ *is* (i) *an amorphous association scheme; or* (ii) *an association scheme in which three of the graphs, say* $\Gamma_2, \Gamma_3, \Gamma_4$*, have the same parameters and which has eigenmatrix given by*

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 & k_4 \\ 1 & s_1 & r_2 & r_3 & r_4 \\ 1 & r_1 & s_2 & s_3 & r_4 \\ 1 & r_1 & s_2 & r_3 & s_4 \\ 1 & r_1 & r_2 & s_3 & s_4 \end{pmatrix},$$

or (iii) *it is not an association scheme, in which case the eigenmatrix is given by*

$$P = \begin{pmatrix} 1 & k_1 & k_2 & k_3 & k_4 \\ 1 & s_1 & s_2 & r_3 & r_4 \\ 1 & s_1 & r_2 & s_3 & r_4 \\ 1 & s_1 & r_2 & r_3 & s_4 \\ 1 & r_1 & s_2 & s_3 & r_4 \\ 1 & r_1 & s_2 & r_3 & s_4 \\ 1 & r_1 & r_2 & s_3 & s_4 \end{pmatrix},$$

where possibly one row is removed.

When we restrict his results to the framework of association schemes, only the cases (i) and (ii) appear.

For the case (ii), van Dam presents the following two examples:

$$P \;=\; \begin{pmatrix} 1 & v-4 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & 1 & -1 & -1 \end{pmatrix}, \tag{1}$$

$$P \;=\; \begin{pmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & 17 & -15 & -15 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & -15 & -15 & 17 \end{pmatrix}. \tag{2}$$

(1) is the wreath product of the complete graph and $L_{1,1,1}(2)$. (2) is constructed as a fusion scheme of the cyclotomic scheme of class $d = 45$ over $GF(2^{12})$. (1) is an imprimitive association scheme, and (2) is a primitive one.

§2. Motivation

The International Conference on Algebraic Combinatorics was held at Vladimir, near Moscow, in 1991. A.V. Ivanov presented the following conjecture:

A.V. Ivanov's Conjecture: Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. We assume that for any $i(\neq 0)$, $\Gamma_i = (X, R_i)$ is a strongly-regular graph. Then $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ is an amorphous association scheme.

Since the examples (1) and (2) exist, the result of Edwin R. van Dam is a counterexample of this conjecture ($d = 4$). We want to investigate A.V. Ivanov's conjecture and want to generalize van Dam's results to larger class d.

We call the following condition *A.V. Ivanov's condition*:

A.V. Ivanov's condition: Let $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ be a symmetric association scheme. For any i $(\neq 0)$, $\Gamma_i = (X, R_i)$ is a strongly-regular graph.

The next question arises naturally:

Question: Classify symmetric association schemes with A.V. Ivanov's condition.

However, in this stage, it is difficult to solve this problem. Later, we supplement one condition about the matrix form P to get one of the generalizations of van Dam's result. In particular, we are interested in the shape of the 1st eigenmatrix P of an association scheme with A.V. Ivanov's condition.

§3. One of the generalizations

From now on, we assume a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0\le i\le d})$ satisfies A.V. Ivanov's condition.

Let P_0 be the right-lower $d \times d$ submatrix of P and call it the *principal part* of P:

$$P = \left(\begin{array}{c|ccc} 1 & k_1 & \cdots & k_d \\ \hline 1 & & & \\ \vdots & & P_0 & \\ 1 & & & \end{array}\right).$$

Note that the rows of P are indexed by the primitive idempotents of the adjacency algebra of the association scheme, and the columns of P are indexed by the set of relations $\{R_i\}_{0\le i\le d}$. Hence, P_0 is determined up to permutations of rows and columns.

From A.V. Ivanov's condition, for any i $(i \neq 0)$ we can make a fusion scheme $\tilde{\mathcal{X}} = (X, \{R_0, R_i, (X \times X) - R_0 - R_i\})$. This fusion scheme must be a strongly regular graph. From this fact, in each column of P_0 there are exactly 2 distinct values a_i and b_i.

Using these facts and permuting the rows and columns suitably, or exchanging a_i and b_i, we may start from the following general form without loss of generality:

$$P = \left(\begin{array}{cc|ccc} 1 & k_1 & k_2 & \cdots & k_d \\ 1 & a_1 & a_2 & \cdots & a_d \\ \vdots & \vdots & & & \\ 1 & a_1 & & & \\ \hline 1 & b_1 & & & \\ \vdots & \vdots & & & \\ 1 & b_1 & & & \end{array}\right).$$

In this paper, we only treat the following matrix form P:

$$P = \left(\begin{array}{ccc|ccc} 1 & k_1 & k_2 & k_3 & \cdots & k_d \\ 1 & a_1 & a_2 & a_3 & \cdots & a_d \\ 1 & b_1 & a_2 & b_3 & \cdots & b_d \\ \hline 1 & b_1 & b_2 & & & \\ \vdots & \vdots & \vdots & & \tilde{T} & \\ 1 & b_1 & b_2 & & & \end{array}\right), \tag{3}$$

where $\tilde{T}_{i,j} \in \{a_j, b_j\}$.

This is an important general form. For this we have two reasons: One is that we investigated symmetric association schemes with A.V. Ivanov's condition with class d, concretely, where $d \leq 6$. Then the matrix form (3) is only left alive as a feasible case. Another is that (3) is *one* of the generalizations of the matrix form given by the paper of van Dam. If we restrict this matrix to the case $d = 4$, we can get van Dam's matrix form (Theorem 1 (ii)). Therefore, we treat this general form.

For the submatrix $\tilde{T}$ in (3), we define the $\{0,1\}$-matrix $T = (T_{i,j})_{\substack{3\leq i\leq d \\ 3\leq j\leq d}}$:

$$T_{i,j} = \begin{cases} 1, & \text{if } \tilde{T}_{i,j} = a_j, \\ 0, & \text{otherwise.} \end{cases}$$

In the next theorem, we mention the combinatorial structure for the incidence matrix T.

Theorem 2 *Let $\mathcal{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a symmetric association scheme with the 1st eigenmatrix* (3)*. Then, we get the followings:*

(i) $k_3 = \ldots = k_d$,

(ii) $m_3 = \ldots = m_d$,

(iii) $a_3 = \ldots = a_d = k_3 a_2 k_2^{-1}$,

(iv) $b_3 = \ldots = b_d = k_3(m_1 a_1 k_1^{-1} + m_2 b_1 k_1^{-1} - m_1 a_2 k_2^{-1})k_2^{-1}$,

(v) *T is the incidence matrix of a symmetric balanced incomplete block design.*

From Theorem 2 (v), since the number of 1 appearing in each row and column in T is independent of the choice of the rows and columns of T, we denote this number by ℓ: For $3 \leq i \leq d$,

$$\ell = \sharp\{j \mid T_{i,j} = 1\}.$$

Proposition 3 *Under the same hypothesis as in Theorem 2, the entries of P and Q are expressed as follows:*

$$m_2 = \frac{(d-\ell-2)\big((d-3)\ell k_1 + (d-2)^2 k_2\big)}{(d-2)^2(d-3)\ell},$$

$$
\begin{aligned}
m_3 &= \frac{(d-3)\ell k_1 + (d-2)^2 k_2}{(d-2)^2 \ell}, \\
k_3 &= \frac{(d-\ell-2)k_2}{(d-3)\ell}, \\
b_1 &= \frac{(d-3)\ell k_1\Big((d-2)^2 + \ell a_1\Big)}{\Big(\ell-(d-2)^2\Big)\Big((d-3)\ell k_1 + (d-2)^2 k_2\Big)}, \\
a_2 &= \frac{(d-3)\ell(a_1+1)}{\ell-(d-2)^2}, \\
b_2 &= -\frac{\ell\Big((d-3)\ell k_1 + (d-\ell-2)(d-2)k_2\Big)a_1 + (d-3)\Big(\ell k_1 - (d-2)^2 k_2\Big)}{\Big(\ell-(d-2)^2\Big)\Big((d-3)\ell k_1 + (d-2)^2 k_2\Big)}, \\
b_3 &= -\frac{(d-3)\Big((d-2)k_2 + \ell k_1\Big)\ell a_1 + (d-3)\ell^2 k_1 - (d-2)^2(d-\ell-2)k_2}{\Big(\ell-(d-2)^2\Big)\Big((d-3)\ell k_1 + (d-2)^2 k_2\Big)}.
\end{aligned}
$$

Moreover, a_1 must satisfy the following equation:

$$s_2 a_1^2 + s_1 a_1 + s_0 = 0,$$

where,

$$
\begin{aligned}
s_2 &= -(d-3)\ell^2\Big(-(d-3)\ell k_1 + \ell k_2 - (d-2)^2 k_2\Big), \\
s_1 &= 2\ell^3(d-3)^2 k_1, \\
s_0 &= -(d-2)^2\Big(\ell-(d-2)^2\Big)k_2^2 \\
&\quad -\ell(d-3)\Big(\ell-(d-2)^2\Big)\Big(-(d-2)^2(k_1+1) + \ell k_1\Big)k_2 + \ell^3(d-3)^2 k_1.
\end{aligned}
$$

Next, we consider the trivial case for $\tilde{T}$, that is,

$$\tilde{T} = a_3 I_{d-2} + b_3(J_{d-2} - I_{d-2}).$$

Then, we get $\ell = 1$. From Theorem 2 and Proposition 3 we get the following:

Proposition 4 *In addition to the assumption of Theorem 2, we assume that in* (3)

$$\tilde{T} = a_3 I_{d-2} + b_3(J_{d-2} - I_{d-2}).$$

Then, $k_2 = k_3$, $a_2 = a_3$ and $b_2 = b_3$. In other words, we have

$$
P = \left(\begin{array}{ccc|cccc}
1 & k_1 & k_2 & k_2 & \cdots & k_2 \\
1 & a_1 & a_2 & a_2 & \cdots & a_2 \\
1 & b_1 & a_2 & b_2 & \cdots & b_2 \\
\hline
1 & b_1 & b_2 & a_2 & & b_2 \\
\vdots & \vdots & \vdots & & \ddots & \\
1 & b_1 & b_2 & b_2 & & a_2
\end{array}\right). \qquad (
$$

Moreover, $n, a_1, b_1, a_2, b_2, k_1, k_2, m_1, m_2$ are expressed as follows:

$$\begin{aligned}
n &= \frac{\left((d-2)^2 m_2 + d - a_2 - 3\right)^2}{a_2{}^2 - 2(d-3)a_2 + (d-3)\left((d-2)^2 m_2 + d - 3\right)}, \\
k_1 &= (d-2)^2 m_1, \\
k_2 &= \frac{(d-2)^2 m_2 a_2}{a_2{}^2 - 2(d-3)a_2 + (d-3)\left((d-2)^2 m_2 + d - 3\right)}, \\
m_1 &= m_2 \cdot \frac{-(d-1)a_2{}^2 - 2a_2 + (d-2)^2 m_2 + d - 3}{a_2{}^2 - 2(d-3)a_2 + (d-3)\left((d-2)^2 m_2 + d - 3\right)}, \\
a_1 &= -(d-1)a_2 - 1, \\
b_1 &= \frac{(d-3-a_2)m_2\left((d-1)a_2{}^2 + 2a_2 - (d-2)^2 m_2 - d + 3\right)}{a_2{}^2 - 2(d-3)a_2 + (d-3)\left((d-2)^2 m_2 + d - 3\right)}, \\
b_2 &= a_2 \cdot \frac{a_2{}^2 - (d-4)a_2 - (d-2)^2 m_2 - d + 3}{a_2{}^2 - 2(d-3)a_2 + (d-3)\left((d-2)^2 m_2 + d - 3\right)},
\end{aligned}$$

where,

$$m_2 > \frac{(d-1)a_2{}^2 + 2a_2 - (d-3)}{(d-2)^2}.$$

If we assume $a_2 = d-3$ and $m_2 = (d-3)s$ $(s \in \mathbf{Z}_{>0})$ in the previous proposition, then (4) reduces to

$$P = \begin{pmatrix}
1 & (d-2)^2(s-1) & d-3 & d-3 & \cdots & d-3 \\
1 & -(d-2)^2 & d-3 & d-3 & \cdots & d-3 \\
1 & 0 & d-3 & -1 & \cdots & -1 \\
1 & 0 & -1 & d-3 & & -1 \\
\vdots & \vdots & \vdots & & & \\
\vdots & \vdots & \vdots & & \ddots & \\
1 & 0 & -1 & -1 & & d-3
\end{pmatrix}. \qquad (5)$$

In (5), (X, R_1) is a complete multipartite graph, and (X, R_i) $(2 \leq i \leq d)$ is of Latin square type $L_1(d-2)$. Therefore, $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ with the 1st eigenmatrix (5) is the wreath product of the complete graph and $L_{\underbrace{1,\ldots,1}_{d-1}}(d-2)$. This is a generalization of

§4. A New Example and Possible Infinite Series

This is a joint work with *Akihiro Munemasa* (Kyushu University). The aim of this section is to present a new example of a non-amorphous association scheme and possible infinite series.

Edwin R. van Dam gives the following infinite series with respect to Theorem 1 (ii):

$$P = \begin{pmatrix} 1 & 2^{3t} - 4 - 3(2^{2t}) - 3(2^t) & 2^{2t} + 2^t + 1 & 2^{2t} + 2^t + 1 & 2^{2t} + 2^t + 1 \\ 1 & -4 - 3(2^t) & 1 + 2^t & 1 + 2^t & 1 + 2^t \\ 1 & -4 + 2^t & 1 + 2^t & 1 - 2^t & 1 - 2^t \\ 1 & -4 + 2^t & 1 - 2^t & 1 + 2^t & 1 - 2^t \\ 1 & -4 + 2^t & 1 - 2^t & 1 - 2^t & 1 + 2^t \end{pmatrix}.$$

In the case $t = 4$, he gives one example of non-amorphous association schemes as a fusion scheme of the cyclotomic scheme of class $d = 45$ over finite field $GF(2^{12})$:

$$P = \begin{pmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & 17 & -15 & -15 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & -15 & -15 & 17 \end{pmatrix}.$$

However, there are no comments for the other values t.

We tried to construct another example in van Dam's infinite series. However, we could not find one !

The next table presents the feasible parameters $n, k_1, k_2, m_1, m_2, a_1, b_1, a_2, b_2$ calculated in Proposition 4 ($d = 4$). (We use the automatic E-mail interface to Professor A.E. Brouwer's database of distance-regular graphs.)

n	k_1	a_1	b_1	Criterion	k_2	a_2	b_2	m_1	m_2	Criterion
512	292	−28	4	∃	73	9	−7	73	146	∃
1728	1256	−40	8		157	13	−11	314	471	
1800	1028	−52	8		257	17	−13	257	514	
4096	3276	−52	12	∃	273	17	−15	819	1092	∃
2160	1016	−64	8		381	21	−15	254	635	
8000	6736	−64	16		421	21	−19	1684	2105	
3872	2212	−76	12		553	25	−19	553	1106	
6348	4616	−76	16		577	25	−21	1154	1731	
3025	1296	−79	9		576	26	−18	324	900	
21952	19512	−88	24		813	29	−27	4878	5691	
4901	2548	−91	13		784	30	−22	637	1421	
3888	1196	−100	8		897	33	−21	299	1196	
6728	3844	−100	16		961	33	−25	961	1922	

n	k_1	a_1	b_1	Criterion	k_2	a_2	b_2	m_1	m_2	Criterion
9600	6620	−100	20		993	33	−27	1655	2648	
15376	12300	−100	24		1025	33	−29	3075	4100	
32768	29596	−100	28		1057	33	−31	7399	8456	
7425	4352	−103	17		1024	34	−26	1088	2112	
13872	10088	−112	24		1261	37	−31	2522	3783	
10693	6804	−115	21		1296	38	−30	1701	2997	
10368	5924	−124	20		1481	41	−31	1481	2962	
30420	25616	−124	32		1601	41	−37	6404	8005	
14801	10000	−127	25		1600	42	−34	2500	4100	
14080	8664	−136	24		1805	45	−35	2166	3971	
25872	20192	−136	32		1893	45	−39	5048	6941	
19845	14036	−139	29		1936	46	−38	3509	5445	
12096	5900	−148	20		2065	49	−35	1475	3540	
14792	8452	−148	24		2113	49	−37	2113	4226	
24300	17672	−148	32		2209	49	−41	4418	6627	
33856	27084	−148	36		2257	49	−43	6771	9028	
25921	19008	−151	33		2304	50	−42	4752	7056	
33125	25012	−163	37		2704	54	−46	6253	8957	
14336	6100	−172	20		2745	57	−39	1525	4270	
20000	11428	−172	28		2857	57	−43	2857	5714	
24276	15536	−172	32		2913	57	−45	3884	6797	
13312	3132	−196	12		3393	65	−39	783	4176	
19360	8604	−196	24		3585	65	−45	2151	5736	
25992	14852	−196	32		3713	65	−49	3713	7426	
27200	14752	−208	32		4149	69	−51	3688	7837	
32768	18724	−220	36		4681	73	−55	4681	9362	
30096	14816	−232	32		5093	77	−55	3704	8797	
23232	7148	−244	20		5361	81	−51	1787	7148	
31740	14936	−244	32		5601	81	−57	3734	9335	
22848	5456	−256	16		5797	85	−51	1364	7161	
27440	9544	−256	24		5965	85	−55	2386	8351	

We introduce the following theorem. This theorem is due to A.E. Brouwer, R.M. Wilson, and Q. Xiang ([4]).

Theorem 5 *(A.E. Brouwer, R.M. Wilson, and Q. Xiang)*
Let p be a prime and $q = p^\kappa$. Let $e|(q-1)$. We assume that there exists $h > 0$ such that

$$h := \ min\ \{h > 0; p^\ell \equiv -1 \ mod\ e\}.$$

We put $t_1 = 2\ell\kappa^{-1}$. Then, for any u_1 $(1 \le u_1 \le e-1)$, strongly-regular graphs with the following eigenvalues exist:

$$k \ = \ \frac{q-1}{e}u_1 \quad \textit{with multiplicity } 1,$$

$$\begin{aligned}\theta_1 &= \frac{u_1}{e}(-1+(-1)^{t_1}\sqrt{q}) \quad \textit{with multiplicity } q-1-k,\\ \theta_2 &= \frac{u_1}{e}(-1+(-1)^{t_1}\sqrt{q})+(-1)^{t_1+1}\sqrt{q} \quad \textit{with multiplicity } k.\end{aligned}$$

If we assume, In Proposition 4 , that the strongly-regular graph (X, R_1) is equal to A.E. Brouwer, R.M. Wilson, and Q. Xiang's strongly-regular graphs, then the parameters e and t_1 in Theorem 5 satisfy the following:

Lemma 1

$$e = \frac{d^2-4d+5}{(d-2)^2}u_1, \qquad t_1 = 1.$$

Lemma 2 *If* $d = 4$, *the size* n *is equal to a power of* 16, *i.e.,* $n = 16^{2h+1}$ $(= 2^{4(2h+1)})$.

Let $n = q^{2h+1}$ and $q = 16$. Under the assumption of Lemma1 and 2, we guess the existence of the following possible infinite series:

$$P = \begin{pmatrix} 1 & 12\frac{q^{2h}-1}{q-1} & \frac{q^{2h}-1}{q-1} & \frac{q^{2h}-1}{q-1} & \frac{q^{2h}-1}{q-1} \\ 1 & \frac{-3q^{h+1}-q+4}{q-1} & \frac{q^{h+1}-1}{q-1} & \frac{q^{h+1}-1}{q-1} & \frac{q^{h+1}-1}{q-1} \\ 1 & q\frac{q^{h}-1}{q-1} & \frac{q^{h+1}-1}{q-1} & -\frac{q^{h+1}-2q^{h}+1}{q-1} & -\frac{q^{h+1}-2q^{h}+1}{q-1} \\ 1 & q\frac{q^{h}-1}{q-1} & -\frac{q^{h+1}-2q^{h}+1}{q-1} & \frac{q^{h+1}-1}{q-1} & -\frac{q^{h+1}-2q^{h}+1}{q-1} \\ 1 & q\frac{q^{h}-1}{q-1} & -\frac{q^{h+1}-2q^{h}+1}{q-1} & -\frac{q^{h+1}-2q^{h}+1}{q-1} & \frac{q^{h+1}-1}{q-1} \end{pmatrix}. \tag{6}$$

We give the reason that (6) is natural infinite series:
If $n = 16$ $(h = 0)$ in (6), then we get the following:

$$P = \begin{pmatrix} 1 & 12 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 1 \end{pmatrix}.$$

This matrix belongs to the matrix form (1).
If $n = 16^3$ $(h = 1)$ in (6), then we get the following:

$$P = \begin{pmatrix} 1 & 3276 & 273 & 273 & 273 \\ 1 & -52 & 17 & 17 & 17 \\ 1 & 12 & 17 & -15 & -15 \\ 1 & 12 & -15 & 17 & -15 \\ 1 & 12 & -15 & -15 & 17 \end{pmatrix}.$$

This matrix is an example given by the paper of van Dam ([5]).
If $n = 16^5$ $(h = 2)$ in (6), then we can construct the following new example as a fusion scheme of the cyclotomic scheme of class $d = 75$ over $GF(2^{20})$ using the computer:

$$P = \begin{pmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 204 & 273 & -239 & -239 \\ 1 & 204 & -239 & 273 & -239 \\ 1 & 204 & -239 & -239 & 273 \end{pmatrix}.$$

This is a new example of non-amorphous association schemes of class $d = 4$.

Acknowledgemnts The author would like to thank Professors Eiichi Bannai and Akihiro Munemasa for several stimulating conversations.

References

[1] E. Bannai, *Subschemes of some association schemes*, J. Algebra, **144** (1991), 167–188.

[2] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, Benjamin/Cummings Publ., Menlo Park, 1984.

[3] A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Heidelberg, 1989.

[4] A.E. Brouwer, R.M. Wilson, and Q. Xiang, *Cyclotomy and strongly regular graphs*, J. Alg. Combin., **10** (1999), 25–28.

[5] Edwin R. van Dam, *Strongly regular decompositions of the complete graph*, J. Alg. Combin., to appear.

[6] T. Ito, A. Munemasa, and M. Yamada, *Amorphous association schemes over the Galois rings of characteristic* 4, Europ. J. Combin., **12** (1991), 513–526.

[7] A.A. Ivanov and C.E. Praeger, *Problem session at ALCOM*-91, Europ. J. Combin., **15** (1994), 105–112.

[8] C.L.M.de Lange, *Some new cyclotomic strongly regular graphs*, J. Alg. Combin., **4** (1995), 329–330.

[9] M.E. Muzychuk, Ph.D.thesis, unpublished.

Kobe Gakuin Women's College
27-1 Hayashiyama-cho, Nagata-ku
Kobe, Japan, 653-0861